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On the dispersion relation of random gravity waves. Part 1. Theoretical framework

By AKIRA MASUDA, YI-YU KUO AND HISASHI MITSUYASU

Research Institute for Applied Mechanics,
Kyushu University, Fukuoka, Japan

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A theoretical framework is given, upon which to examine the dispersion relation of random gravity waves. First a weakly nonlinear theory is developed to the third-order for a statistically stationary and homogeneous field of random gravity waves. Both the spectrum of forced waves and the nonlinear dispersion relation are expressed in terms of the spectrum of free waves under the assumption of the Gaussian process for the first-order surface displacement. Next a method is proposed by which to separate each of the spectra of free and forced waves from the measured spectrum. This gives practical and powerful means of investigating the statistical structure of wind waves.

1. Introduction

To a first approximation, ocean waves can be described as a superposition of statistically independent free waves which have random phases and satisfy the linear dispersion relation. Obviously the dispersion relation is a keystone in the investigation of waves. For example, transformation from the spectrum of wave-slope to that of the surface displacement is totally dependent on the dispersion relation.

Recently, however, several laboratory experiments have brought up suspicions against an approximation of the linear dispersion relation in wind–wave fields. The observed dispersion relation deviated far from the linear one for frequencies near to or higher than twice the spectral peak frequency, which suggests the existence of nonlinear characteristics in wind–wave fields.

Among studies on nonlinear random surface waves, those of Tick (1959), Phillips (1960), Hasselmann (1962), Longuet-Higgins & Phillips (1962), Huang & Tung (1976), Weber & Barrick (1977) and Barrick & Weber (1977) are related to our present work. They have discussed the nonlinear spectrum or the nonlinear dispersion relation. In their studies, however, the dispersion relation has not been formulated to the extent that it may be directly compared with observations. So, the present paper is intended to provide a theoretical framework on which to examine the dispersion relation of measured wind waves. This framework is composed of two parts: first, a weakly nonlinear theory to the third order with respect to the surface displacement (to the second order with respect to the energy spectrum); secondly, a shorter but practically important part, a method to separate each of the spectra for free and forced waves from the measured spectrum.

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The last three of the aforementioned studies are referred to briefly here, since they have treated the same subjects as the present investigation. The paper of Huang & Tung is a stimulating one. Unfortunately, however, the derivation of equation (14) or (15) in Huang & Tung appeaιs to be erroneous since the Fourier–Stiltjes coefficient is taken as a function of both position and wavenumber. Although the results of Weber & Barrick and Barrick & Weber are very similar to ours, it is easy to see that their formulation and ours are different in many aspects. In particular they start with deterministic linear waves in limited areas whereas we consider stochastic properties in infinite regions and the perturbation of linear waves is not used from the beginning. Consequently our derivation takes quite different form from theirs. Moreover our method to separate free and forced waves is completely new and clear. Without this method and idea, comparisoy of theory with observation is impossible.

2. Formulation

We formulate the problem along the same line as that of Phillips (1960) except that the time differentiation is replaced by the multiplication of $(-i\omega)$, where ω is the angular frequency.

Consider a field of deep-water random waves, statistically stationary and homogeneous. Since we consider irrotational motion of an incompressible fluid there exists a velocity potential $\Phi(\mathbf{x}, z, t)$ by which the velocity vector $\mathbf{V}$ is expressed as $\mathbf{V} = \nabla\Phi$ Here $\mathbf{x} = (x, y)$ are the horizontal co-ordinates, z the vertical one (positive upward) and t is the time. The equation of continuity becomes

$$\nabla^2\Phi = 0. \tag{2.1}$$

The solution of (2.1) with the condition that Φ vanishes as $z \to -\infty$ is given formally by

$$\Phi = \int_K dA(K) \exp(i\chi + |\mathbf{k}|z), \tag{2.2}$$

where K denotes $(\omega, \mathbf{k})$, $\mathbf{k}$ being the wavenumber vector; the increment $dA(K)$ is a random variable of $K = (\omega, \mathbf{k})$ and $\chi \equiv \mathbf{k}.\mathbf{x} - \omega t$ is the phase. This is the Fourier–Stieltjes representation of the velocity potential where the integration is over the entire wavenumber-frequency space. In the same way the surface displacement η can be expressed as

$$\eta = \int_K dB(K)\, e^{i\chi}. \tag{2.3}$$

The kinematic and dynamic boundary conditions to be satisfied by Φ and ζ at the surface are

$$\partial\eta/\partial t + \nabla\Phi \,.\, \nabla\eta = \partial\Phi/\partial z \quad \text{at} \quad z = \eta, \tag{2.4}$$

and

$$\partial\Phi/\partial t + \tfrac{1}{2}(\nabla\Phi)^2 = -\eta \quad \text{at} \quad z = \eta, \tag{2.5}$$

where the acceleration due to gravity is taken as unity for convenience.

For the purpose of eliminating $dA(K)$ and obtaining the equation for $dB(K)$, we substitute (2.2) and (2.3) into (2.4) and (2.5) and expand the equations formally following the usual procedure (see Phillips 1960). For example, we have

$$\left.\begin{aligned}(\nabla\Phi)_\eta \,.\, \nabla\eta &= \int_K\int_{K_1} dA(K)\,dB(K_1)\exp[i(\chi+\chi_1)+|\mathbf{k}|\eta](-\mathbf{k}\,.\,\mathbf{k}_1)\\ &= \int_K\int_{K_1} dA(K)\,dB(K_1)\exp[i(\chi+\chi_1)](-\mathbf{k}\,.\,\mathbf{k}_1)(1+|\mathbf{k}|\eta+\tfrac{1}{2}|\mathbf{k}|^2\eta^2+\ldots)\\ &= \int_K\int_{K_1} dA(K)\,dB(K_1)\exp[i(\chi+\chi_1)](-\mathbf{k}\,.\,\mathbf{k}_1)\left(1+\int_{K_2}|\mathbf{k}|\,dB(K_2)\exp(i\chi_2)+\ldots\right).\end{aligned}\right\} \quad (2.6)$$

After a simple though lengthy manipulation we obtain the equation for $dB(K)$ to the third order:

$$\begin{aligned}W(K)\,dB(K) &= \int_{K_1} f_2(K,K_1)\,dB(K_1)\,dB(K-K_1)\\ &\quad+\int_{K_1}\int_{K_2} f_3(K,K_1,K_2)\,dB(K_1)\,dB(K_2)\,dB(K-K_1-K_2),\end{aligned} \quad (2.7)$$

where

$$W(K) = 1-\omega^2/|\mathbf{k}|, \quad (2.8)$$

$$\begin{aligned}&f_2(K,K_1)\\ &\quad= \tfrac{1}{2}\{\omega_1^2-\omega_1\omega+\omega^2-\omega_1(\omega-\omega_1)\langle\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1\rangle-\omega\omega_1\langle\mathbf{k},\mathbf{k}_1\rangle-\omega(\omega-\omega_1)\langle\mathbf{k},\mathbf{k}-\mathbf{k}_1\rangle\}\end{aligned} \quad (2.9)$$

and

$$\left.\begin{aligned}f_3(K,K_1,K_2) = {}&\tfrac{1}{4}(\omega_1^2|\mathbf{k}_1|+\omega_2^2|\mathbf{k}_2|)-\tfrac{1}{4}(\omega\omega_1|\mathbf{k}_1|+\omega\omega_2|\mathbf{k}_2|)\\ &-\tfrac{1}{2}(\omega_1+\omega_2)\,|\mathbf{k}_1+\mathbf{k}_2|\,\{\omega_1\langle\mathbf{k}_1,\mathbf{k}_1+\mathbf{k}_2\rangle+\omega_2\langle\mathbf{k}_2,\mathbf{k}_1+\mathbf{k}_2\rangle\}\\ &-\tfrac{1}{2}\omega(\omega_1+\omega_2)\frac{|\mathbf{k}_1|\,|\mathbf{k}_2|}{|\mathbf{k}|}\langle\mathbf{k}_1,\mathbf{k}_2\rangle\\ &+\tfrac{1}{2}\omega|\mathbf{k}_1+\mathbf{k}_2|\,\langle\mathbf{k},\mathbf{k}_1+\mathbf{k}_2\rangle(\omega_1\langle\mathbf{k}_1,\mathbf{k}_1+\mathbf{k}_2\rangle+\omega_2\langle\mathbf{k}_2,\mathbf{k}_1+\mathbf{k}_2\rangle).\end{aligned}\right\} \quad (2.10)$$

Here $\langle\mathbf{k},\mathbf{k}'\rangle \equiv \mathbf{k}\,.\,\mathbf{k}'/|\mathbf{k}|\,|\mathbf{k}'|$ denotes the cosine of the angle formed by two vectors $\mathbf{k}$ and $\mathbf{k}'$.

Note that some symmetrizations are made in the above derivation for the later computational convenience, so that we easily have

$$f_2(0,K) = f_2(K,0) = 0, \quad (2.11)$$

$$f_2(K,K_1) = f_2(K,K-K_1) = f_2(K_1,K), \quad (2.12)$$

and

$$f_3(K,K_1,K_2) = f_3(K,K_2,K_1). \quad (2.13)$$

Equation (2.7) is the one required; in particular for infinitesimal waves, it states that $W(K)$ must vanish so that $dB(K)$ may not. That is, for such K that $dB(K) \neq 0$ the linear dispersion relation $W(K) = 0$ must be satisfied.

For further development we assume that

$$\eta = \int_K dB_1(K)+\int_K dB_2(K)+\int_K dB_3(K)+\ldots, \quad (2.14)$$

where $\int_K dB_1(K)$ denotes the first-order free waves and so on. Next we assume the

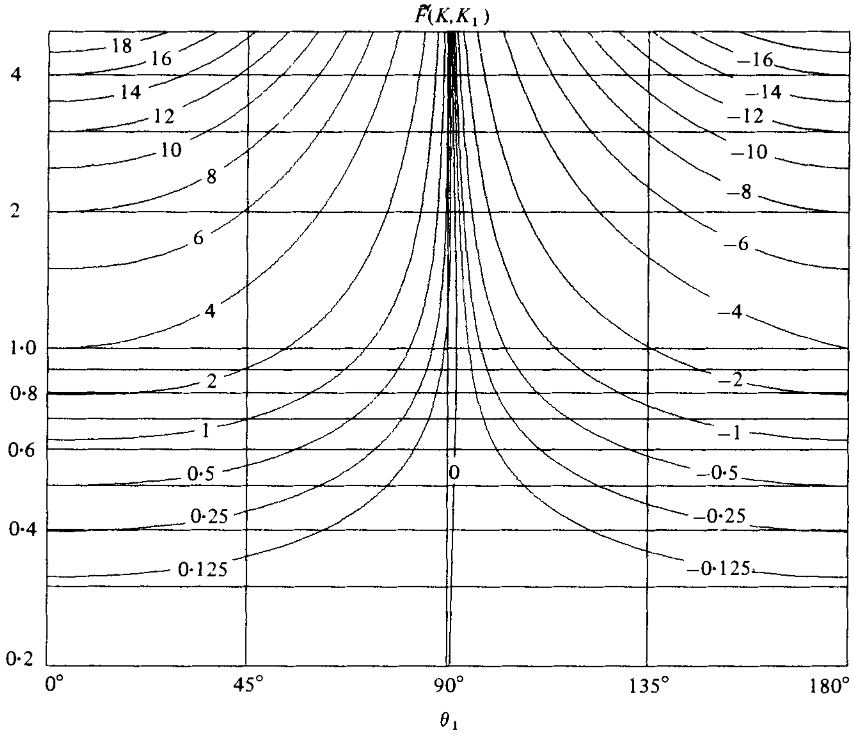

FIGURE 1. Contours of the kernel $\tilde{F}(K, K_1)$, where $K = (\omega, \mathbf{k}) = \{\omega, |\mathbf{k}|, \theta\}$ is assigned as $\{1, 1, 0\}$ while $K_1 = \{\omega_1, |\mathbf{k}_1|, \theta_1\}$ is varied. The free wave K_1 is represented by two quantities, i.e. the frequency ω_1 and the direction θ_1 of propagation, because $|\mathbf{k}_1|$ is determined from the dispersion relation.

Gaussian process for η_1, as Tick (1959) or Hasselmann (1962) has done. Then we have

$$\overline{\left(\frac{dB_1(K)\, dB_1(K')}{dK\, dK'}\right)} = \phi_1(K)\, \delta(K+K'), \tag{2.15}$$

$$\overline{\left(\frac{dB_1(K)\, dB_1(K')\, dB_1(K'')}{dK\, dK'\, dK''}\right)} = 0 \tag{2.16}$$

and

$$\overline{\left(\frac{dB_1(K)\, dB_1(K')\, dB_1(K'')\, dB_1(K''')}{dK\, dK'\, dK''\, dK'''}\right)}$$
$$= \phi_1(K)\, \phi_1(K'')\, \delta(K+K')\, \delta(K''+K''') + \phi_1(K)\, \phi_1(K'')\, \delta(K+K''')\, \delta(K'+K'')$$
$$+ \phi_1(K)\, \phi_1(K')\, \delta(K+K'')\, \delta(K'+K'''), \tag{2.17}$$

where the overbar denotes ensemble average, $\delta(K)$ the delta function and $dK = d\omega\, d\mathbf{k}$.

For $dB(K)$, the assumption of statistical stationarity and homogeneity gives

$$\overline{\left(\frac{dB(K)\, dB(K')}{dK\, dK'}\right)} = \phi(K)\, \delta(K+K'). \tag{2.18}$$

Multiplying (2.7) by $\int_{K_4} dB(K_4)$ and taking ensemble average with the use of (2.11)

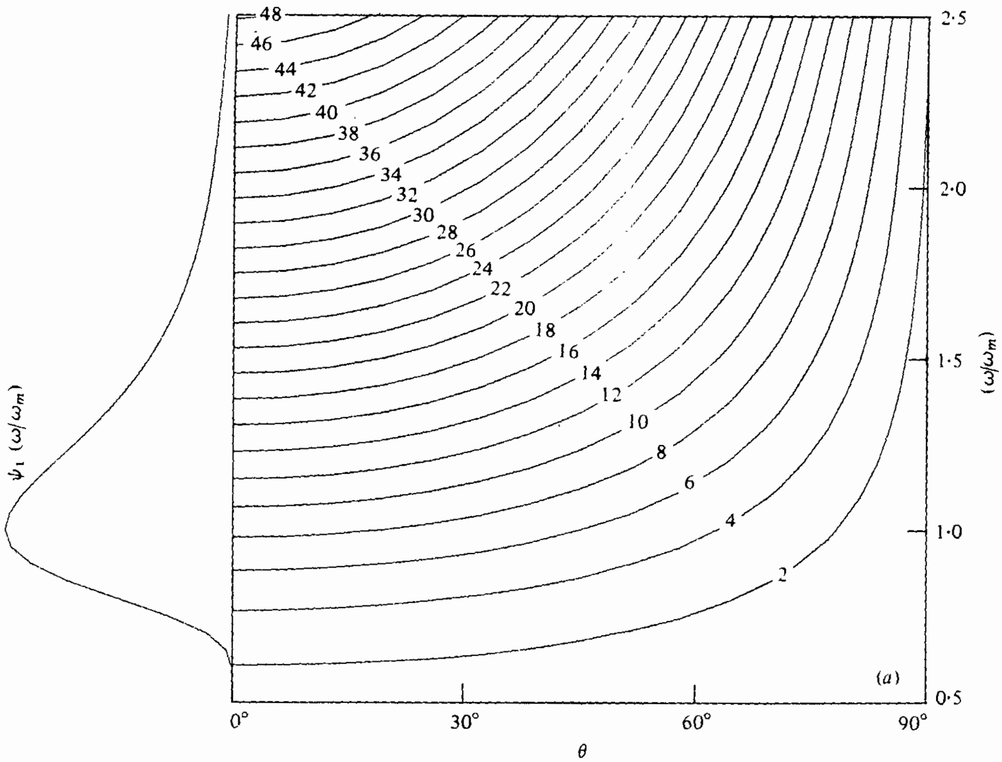

FIGURE 2 (*a*). For legend see next page.

through (2.18) we finally obtain

$$W(K)\,\phi(K) = \int_{K_1} \left\{\frac{G(K,K_1)}{W(K)}\right\} \phi_1(K_1)\,\phi_1(K-K_1)\,dK_1 - \left(\int_{K_1} F(K,K_1)\,\phi_1(K_1)\,dK_1\right)\phi_1(K), \quad (2.19)$$

where $$G(K,K_1) = 2\{f_2(K,K_1)\}^2; \quad (2.20)$$

and

$$F(K,K_1) = -\frac{4\{f_2(K,K_1)\}^2}{W(K-K_1)} - f_3(K,K_1,-K_1) - f_3(K,K_1,K) - f_3(K,K,K_1). \quad (2.21)$$

These are the basic equations for the following analysis. We shall firstly investigate free waves and their nonlinear dispersion relation and then forced waves.

(1) *Free waves*. Free waves here are defined as those which can have non-zero $\phi_1(K)$. We reasonably assume that free waves approximately satisfy the linear dispersion relation. In other words, free waves exist only near $W(K) = 0$. Then we have from (2.19)

$$\left\{W(K) + \int_{K_1} F(K,K_1)\,\phi_1(K_1)\,dK_1\right\}\phi(K) = \int_{K_1} \frac{G(K,K_1)}{W(K)}\,\phi_1(K_1)\,\phi_1(K-K_1)\,dK_1. \quad (2.22)$$

The product of $\phi_1(K_1)$ and $\phi_1(K-K_1)$ vanishes except when $W(K_1) \doteqdot W(K-K_1) \doteqdot 0$, which is incompatible with the condition $W(K) \doteqdot 0$. Therefore, the right-hand side of

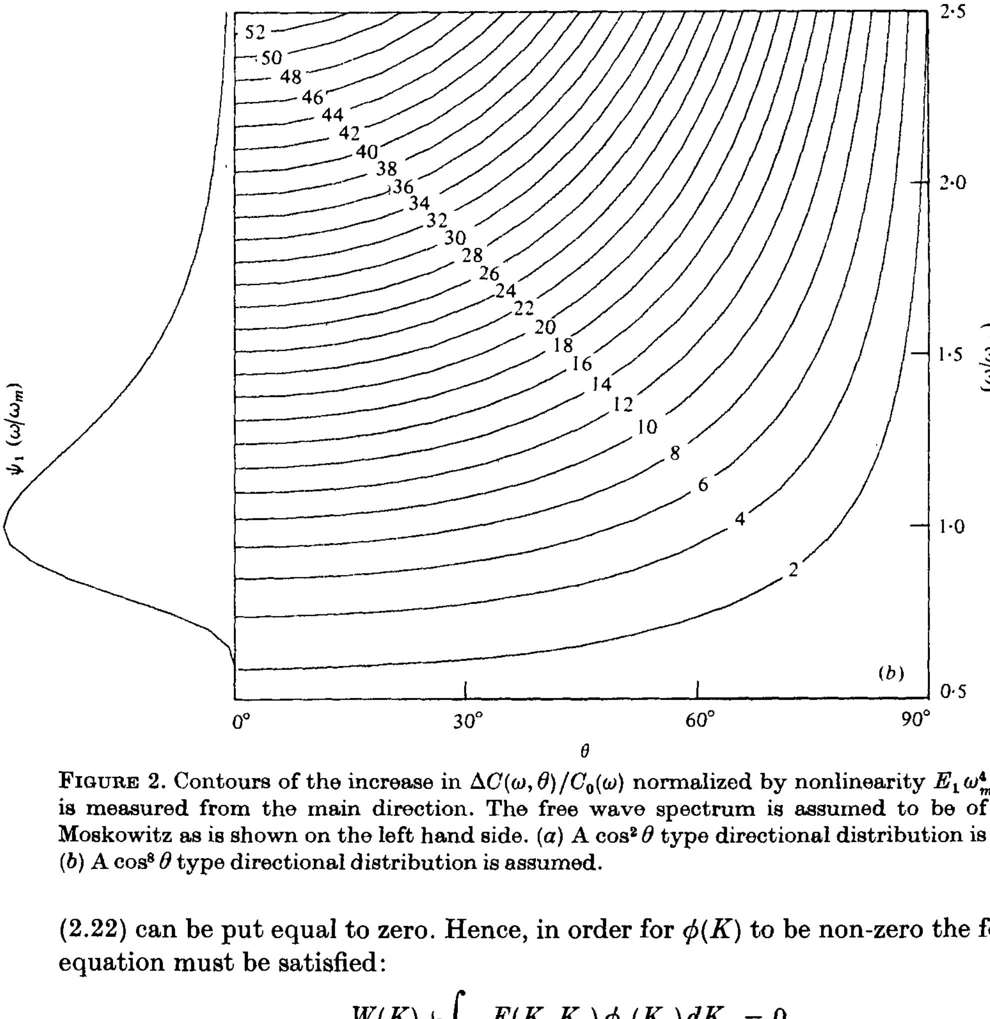

FIGURE 2. Contours of the increase in $\Delta C(\omega, \theta)/C_0(\omega)$ normalized by nonlinearity $E_1 \omega_m^4$, where θ is measured from the main direction. The free wave spectrum is assumed to be of Pierson–Moskowitz as is shown on the left hand side. (*a*) A $\cos^2\theta$ type directional distribution is assumed. (*b*) A $\cos^8\theta$ type directional distribution is assumed.

(2.22) can be put equal to zero. Hence, in order for $\phi(K)$ to be non-zero the following equation must be satisfied:

$$W(K) + \int_{K_1} F(K, K_1)\,\phi_1(K_1)\,dK_1 = 0. \tag{2.23}$$

Alternatively equation (2.23) can be expressed as

$$\Delta C(K)/C_0(K) = \int_{K_1} F(K, K_1)\,\phi_1(K_1)\,dK_1, \tag{2.24}$$

where $$\Delta C(K) \equiv C(K) - C_0(K) = \omega/|\mathbf{k}| - 1/\omega$$

and $C_0(K) = 1/\omega$ is the phase velocity of linear waves. Thus we may interpret (2.23) or (2.24) as a nonlinear dispersion relation for weakly nonlinear random waves. In fact for a uni-directional wave field, (2.24) may be expressed as

$$\Delta C(\omega)/C_0(\omega) = \int_0^{\omega} 8\omega_1^3\,\omega\phi_1(\omega_1)\,d\omega_1 + \int_{\omega}^{\infty} 8\omega_1\,\omega^3\phi_1(\omega_1)\,d\omega_1, \tag{2.25}$$

which is in agreement with the result of Longuet-Higgins & Phillips (1962) (see appendix).

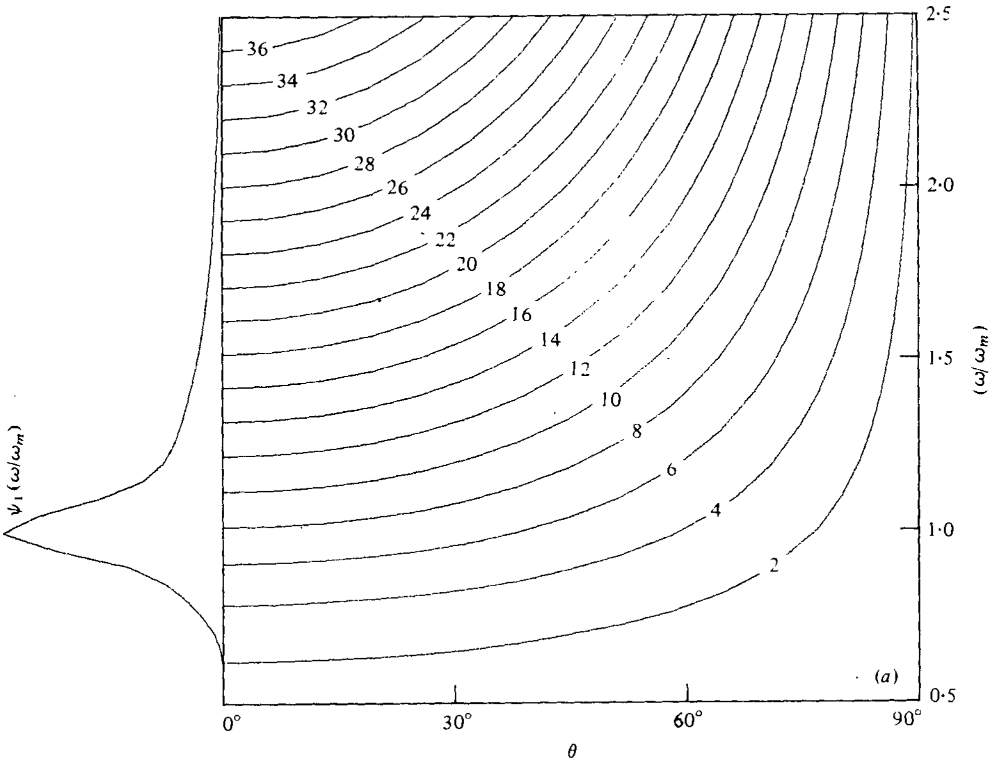

FIGURE 3(*a*). For legend see next page.

The kernel $F(K, K_1)$ denotes the magnitude of the contribution to $\Delta C/C_0$ from free waves K_1. More conveniently it can be rewritten as

$$\begin{aligned} \tilde{F}(K,K_1) &\equiv \tfrac{1}{2}\{F(K,K_1)+F(K,-K_1)\} \\ &= -2\left(\frac{\{f_2(K,K_1)\}^2}{W(K-K_1)}+\frac{\{f_2(K,-K_1)\}^2}{W(K+K_1)}\right)+(\omega_1^2|\mathbf{k}_1|+\omega^2|\mathbf{k}|) \\ &\quad +2\omega\,\omega_1(|\mathbf{k}|+|\mathbf{k}_1|)\langle\mathbf{k},\mathbf{k}_1\rangle \\ &\quad -\tfrac{1}{2}|\mathbf{k}+\mathbf{k}_1|\{\omega\langle\mathbf{k},\mathbf{k}+\mathbf{k}_1\rangle+\omega_1\langle\mathbf{k}_1,\mathbf{k}+\mathbf{k}_1\rangle\}^2 \\ &\quad -\tfrac{1}{2}|\mathbf{k}-\mathbf{k}_1|\{\omega\langle\mathbf{k},\mathbf{k}-\mathbf{k}_1\rangle+\omega_1\langle\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1\rangle\}^2. \end{aligned} \tag{2.26}$$

In figure 1 the contours of $\tilde{F}(K, K_1)$ are shown when $\omega = 1$, $|\mathbf{k}| = 1$ and the propagation direction $\theta = 0°$. We can see that the phase velocity of the wave K increases owing to the nonlinear effects from other waves propagating in the direction within a certain angle slightly more than 90°. On the other hand, waves propagating in the opposite direction contribute to decrease the phase velocity. Note that setting a particular K means no loss of generality. For, if necessary, $K_1 = \{\omega_1, |\mathbf{k}_1|, \theta_1\}$ is to be read as $\{\omega_1/\omega, |\mathbf{k}_1|/|\mathbf{k}|, \theta_1-\theta\}$, where $\{\omega, |\mathbf{k}|, \theta\}$ denotes the polar co-ordinate representation of $\mathbf{k}$.

In order to illustrate the increase in phase velocity due to nonlinearity figures 2 and 3 are presented, where Pierson–Moskowitz and JONSWAP spectra are adopted as typical examples. Assuming these spectra to be of free waves with directional distribution of $\cos^P\theta$ type, we can calculate $\Delta C(\omega,\theta)/C_0(\omega)$ normalized by the strength

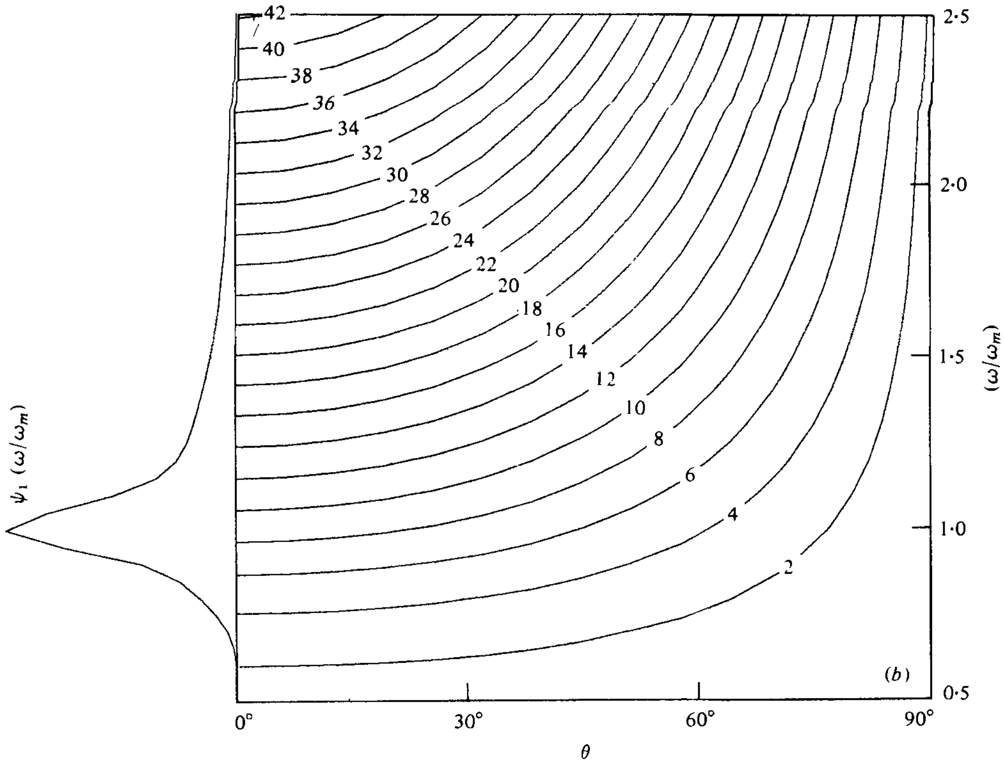

FIGURE 3. Contours as in figure 2 but for JONSWAP spectra. (a) A $\cos^2\theta$ type directional distribution is assumed. (b) A $\cos^8\theta$ type directional distribution is assumed.

of nonlinearity $E_1\omega_m^4$, where E_1 is the total energy of free waves and ω_m the peak frequency. The figures show the effects of the spectral form and directional distribution on $\Delta C/C_0$. We can see, for example, that the more gradual the spectral form (Pierson–Moskowitz type) and the more concentrated the directional distribution, the greater the increase in phase velocity expected, if nonlinearity $E_1\omega_m^4$ is held constant. These features are easily understood from the properties of the kernel function $\tilde{F}(k, k_1)$ displayed in figure 1.

(2) *Forced waves.* For K not close to $W(K) = 0$ we find forced waves:

$$\phi_2(K) = \int_{K_1} \tilde{G}(K, K_1)\,\phi_1(K_1)\,\phi_1(K-K_1)\,dK_1, \tag{2.27}$$

where

$$\tilde{G}(K, K_1) \equiv \frac{G(K, K_1)}{W^2(K)} = 2\left\{\frac{f_2(K, K_1)}{W(K)}\right\}^2. \tag{2.28}$$

These forced waves are merely accompaniments of free waves. In many cases of wind-generated waves, however, the frequency spectrum of forced waves, $\int \phi_2(\omega, \mathbf{k})\,dk$ seems to prevail over that of free waves $\int \phi_1(\omega, \mathbf{k})\,d\mathbf{k}$, for frequencies close to twice the spectral peak frequency. This fact may be of crucial importance when the phase velocity is determined experimentally by the cross-spectrum method.

Figures 4–6 are presented to show in detail the generation of forced waves. The forced wave is expressed as

$$K = (\omega, \mathbf{k}) = K_1 + K_2 = (\omega_1 + \omega_2, \mathbf{k}_1 + \mathbf{k}_2).$$

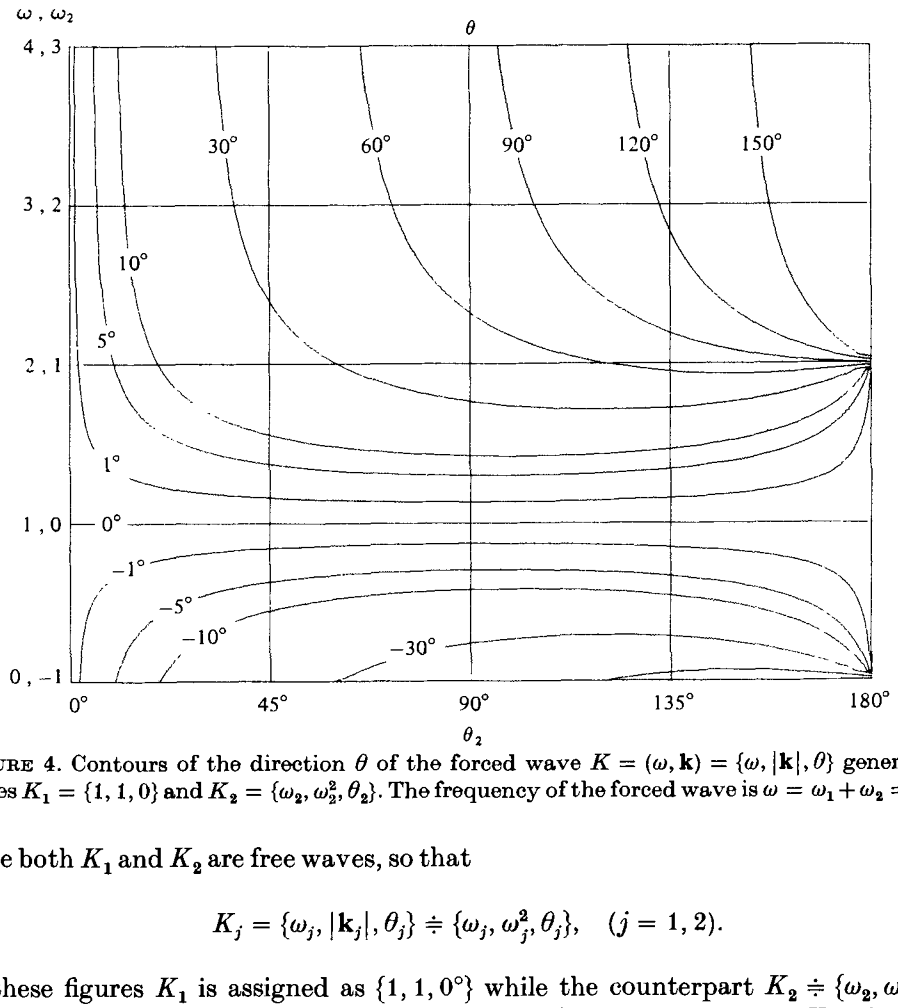

FIGURE 4. Contours of the direction θ of the forced wave $K = (\omega, \mathbf{k}) = \{\omega, |\mathbf{k}|, \theta\}$ generated by waves $K_1 = \{1, 1, 0\}$ and $K_2 = \{\omega_2, \omega_2^2, \theta_2\}$. The frequency of the forced wave is $\omega = \omega_1 + \omega_2 = 1 + \omega_2$.

Here both K_1 and K_2 are free waves, so that

$$K_j = \{\omega_j, |\mathbf{k}_j|, \theta_j\} \doteqdot \{\omega_j, \omega_j^2, \theta_j\}, \quad (j = 1, 2).$$

In these figures K_1 is assigned as $\{1, 1, 0°\}$ while the counterpart $K_2 \doteqdot \{\omega_2, \omega_2^2, \theta_j\}$ is varied. Figures 4 and 5 respectively show θ and $|\mathbf{k}|$ of the forced wave $K = \{\omega, |\mathbf{k}|, \theta\}$. Of course, the frequency ω equals to ω_1 plus ω_2. From figure 5 we observe that forced waves cannot satisfy the linear dispersion relation ($\omega^2 = |\mathbf{k}|$) except the trivial case $\omega_2 = 0$. Figure 6 shows contours of $\tilde{G}(K, K_1)$ which indicate the magnitude of the contribution to the spectrum of forced waves $\phi_2(K)$ from free waves K_1 and K_2. Similarly to figure 1, the choice of a particular K_1 leads to no loss of generality in these figures.

With these preparations, we have the correlation $R(\mathbf{l}, \tau)$ of the surface displacements of two points separated by a distance l:

$$\left.\begin{aligned} R(\mathbf{l},\tau) &\equiv \overline{\zeta(\mathbf{x},t)\,\eta(\mathbf{x}+\mathbf{l},t+\tau)} \\ &= \int_\omega\int_\mathbf{k} \exp(-i\mathbf{k}.\mathbf{l}+i\omega t)\,\phi(\omega,\mathbf{k})\,d\mathbf{k}\,d\omega \\ &= \int_\omega C_r(\omega,\mathbf{l})\,e^{i\omega\tau}\,d\omega, \end{aligned}\right\} \tag{2.29}$$

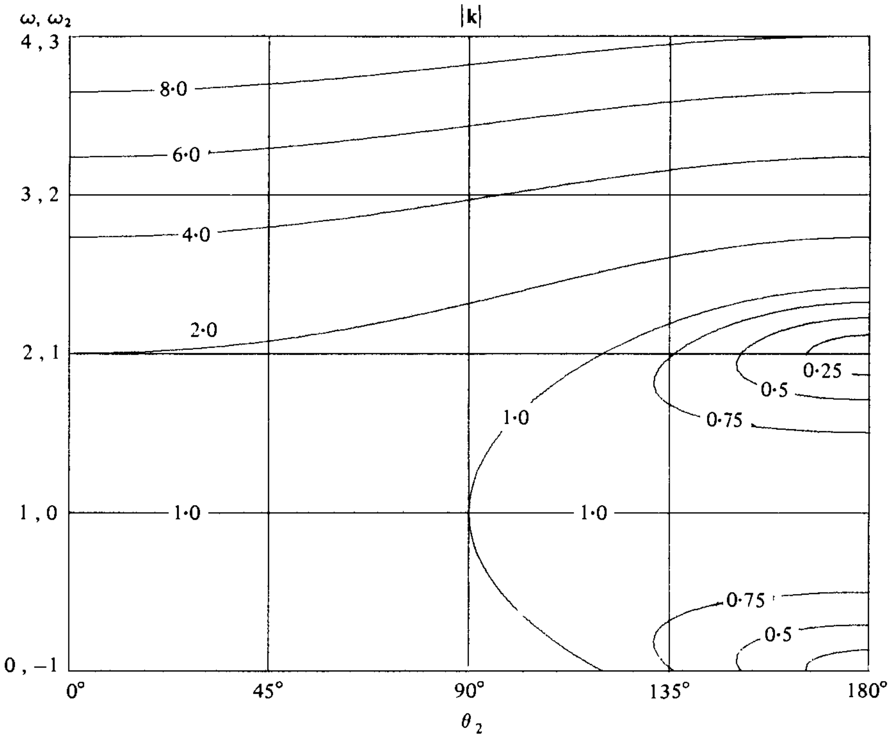

FIGURE 5. Contours of the magnitude of wavenumber $|\mathbf{k}|$ of the forced wave. For detailed legend see figure 4.

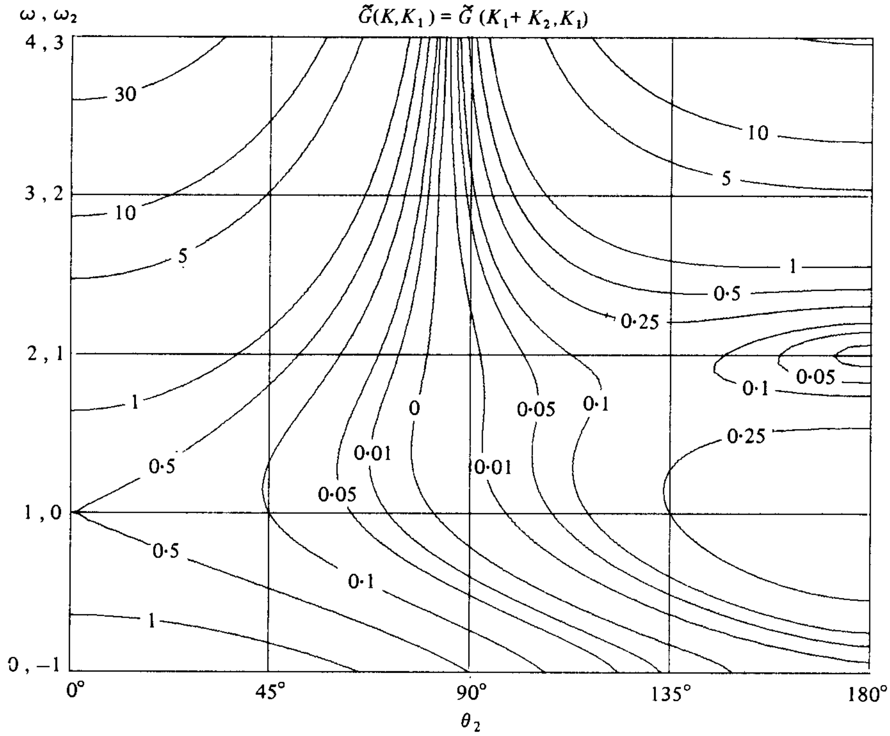

FIGURE 6. Contours of the kernel $\tilde{G}(K, K_1)$. For detailed legend see figure 4.

where τ is a time delay and $C_r(\omega, \mathbf{l})$ is the cross spectrum:

$$C_r(\omega,\mathbf{l}) = \int_{\mathbf{k}} \exp[-i\mathbf{k}.\mathbf{l}]\,\phi(\omega,\mathbf{k})\,d\mathbf{k} = \int_{\mathbf{k}} \exp[-i\mathbf{k}.\mathbf{l}]\,\{\phi_1(\omega_1,\mathbf{k}) + \phi_2(\omega,\mathbf{k})\}\,d\mathbf{k} = C_{r_1}(\omega,\mathbf{l}) + C_{r_2}(\omega,\mathbf{l}). \quad (2.30)$$

In the actual computation of $C_{r_2}(\omega,\mathbf{l})$ and $\psi_2(\omega) = C_{r_2}(\omega, 0)$ it is better to integrate over $\mathbf{k}_2 \equiv \mathbf{k}-\mathbf{k}_1$ rather than $\mathbf{k}$. That is, to use

$$C_{r_2}(\omega,\mathbf{l}) = \int_{\mathbf{k}_2}\int_{\mathbf{k}_1}\int_{\omega_1} \exp[-i(\mathbf{k}_1+\mathbf{k}_2).\mathbf{l}]\,\tilde{G}((\omega,\mathbf{k}_1+\mathbf{k}_2),(\omega_1,\mathbf{k}_1))\,\phi_1(\omega_1,\mathbf{k}_1) \times \phi_1(\omega-\omega_1,\mathbf{k}_2)\,d\omega_1\,d\mathbf{k}_1\,d\mathbf{k}_2, \quad (2.31)$$

and

$$\psi_2(\omega) = \int_{\mathbf{k}_2}\int_{\mathbf{k}_1}\int_{\omega_1} \tilde{G}((\omega,\mathbf{k}_1+\mathbf{k}_2),(\omega_1,\mathbf{k}_1))\,\phi_1(\omega_1,\mathbf{k}_1)\,\phi_1(\omega-\omega_1,\mathbf{k}_2)\,d\omega_1\,d\mathbf{k}_1\,d\mathbf{k}_2. \quad (2.32)$$

3. Separation of the spectrum of forced waves from measured spectrum

To apply the preceding theory to the analysis of the observed waves it is necessary to know $\phi_1(K)$ and $\phi_2(K)$. Since the measured spectrum of the surface displacement is the result of both free waves and forced waves, we must separate it in a reasonable way. Fortunately, the separation is possible to a rough extent by the following iterative method.

Within our present framework we consider that

$$\psi_{\text{obs}}(\omega) = \psi_1(\omega) + \psi_2(\omega), \quad (3.1)$$

where $\psi_{\text{obs}}(\omega)$ is the frequency spectrum measured, while $\psi_1(\omega)$ and $\psi_2(\omega)$ are those of free and forced waves respectively. The frequency spectra $\psi_i(\omega)$ $(i = 1, 2)$ are related to $\phi_i(K)$ by

$$\psi_i(\omega) = \int_{\mathbf{k}} \phi_i(\omega,\mathbf{k})\,d\mathbf{k}. \quad (3.2)$$

Since we do not possess such reliable information about the directional distribution, we assume a suitable form as $S(\omega,\theta)$. Then we have the spectrum of free waves:

$$\phi_1(\omega,\mathbf{k})\,d\omega\,d\mathbf{k} = \psi_1(\omega)\,S(\omega,\theta)\,\delta(|\mathbf{k}| - \omega^2 + \epsilon(\omega,\theta))\,d\omega\,d|\mathbf{k}|\,d\theta, \quad (3.3)$$

where $\epsilon(\omega,\theta)$ is an increment of $|\mathbf{k}|$ deduced by the non-linear dispersion relation (2.23). The spectrum of forced waves $\phi_2(K)$ is given by (2.27):

$$\phi_2(\omega,\mathbf{k})\,d\omega\,d\mathbf{k} = d\omega\,d\mathbf{k}\int_{K_1} \tilde{G}(K,K_1)\,\phi_1(K_1)\,\phi_1(K-K_1)\,dK_1. \quad (3.4)$$

The problem is to solve (3.1) through (3.4) on the data of $\psi_{\text{obs}}(\omega)$. We put the following approximations at zero:

$$\psi_2^{(0)}(\omega) = 0, \quad \phi_2^{(0)}(K)\,dK = 0, \quad \psi_1^{(0)}(\omega) = \psi_{\text{obs}}(\omega), \quad (3.5)\text{–}(3.7)$$

and

$$\phi_1^{(0)}(K)\,dK = \psi_1^{(0)}(\omega)\,S(\omega,\theta)\,\delta(|\mathbf{k}| - \omega^2)\,d\omega\,d|\mathbf{k}|\,d\theta, \quad (3.8)$$

where an overscript in parenthesis denotes the order of iteration. The first and second approximations ($j = 1, 2$) are

$$\phi_2^{(j)}(K)\,dK = dK\int_{K_1}\tilde{G}(K,K_1)\,\phi_1^{(j-1)}(K_1)\,\phi_1^{(j-1)}(K-K_1)\,dK_1, \tag{3.9}$$

$$\psi_2^{(j)}(\omega) = \int_{\mathbf{k}}\phi_2^{(j)}(\omega,\mathbf{k})\,d\mathbf{k}, \tag{3.10}$$

$$\psi_1^{(j)}(\omega) = \psi_{\text{obs}}(\omega) - \psi_2^{(j)}(\omega). \tag{3.11}$$

and

$$\phi_1^{(j)}(K)\,dK = d\omega\, d|\mathbf{k}|\, d\theta\,\psi_1^{(j)}(\omega)\, S(\omega,\theta)\,\delta(|\mathbf{k}|-\omega^2). \tag{3.12}$$

A preliminary computation for our data of wind-generated waves showed that this iterative method yields very rapid convergence and that the second approximations are sufficient for our present purpose of studying the dispersion relation.

In order to find the nonlinear dispersion relation, we use (2.24):

$$\Delta C(K)/C_0(K) = \int_{K_1}\tilde{F}(K,K_1)\,\phi_1^{(2)}(K_1)\,dK_1 \tag{3.13}$$

$\epsilon(\omega,\theta)$ is calculated from (3.13) and $\phi_1^{(2)}(K)$ is modified to give

$$\phi_1^{(2\cdot5)}(K)\,dK = d\omega\, d|\mathbf{k}|\, d\theta\,\psi_1^{(2)}(\omega)\, S(\omega,\theta)\,\delta(|\mathbf{k}|-\omega^2+\epsilon(\omega,\theta)). \tag{3.14}$$

The third-order approximations associated with this change become

$$\phi_2^{(3)}(K)\,dK = dK\int_{K_1}\tilde{G}(K,K_1)\,\phi_1^{(2\cdot5)}(K_1)\,\phi_1^{(2\cdot5)}(K-K_1)\,dK_1, \tag{3.15}$$

$$\psi_2^{(3)}(\omega) = \int_{\mathbf{k}}\phi_2^{(3)}(\omega,\mathbf{k})\,d\mathbf{k}, \tag{3.16}$$

$$\psi_1^{(3)}(\omega) = \psi_{\text{obs}}(\omega) - \psi_2^{(3)}(\omega), \tag{3.17}$$

and

$$\phi_1^{(3)}(K)\,dK = d\omega\, d|\mathbf{k}|\, d\theta\,\psi_1^{(3)}(\omega)\, S(\omega,\theta)\,\delta(|\mathbf{k}|-\omega^2+\epsilon(\omega,\theta)). \tag{3.18}$$

4. Discussion

It appears that the present theory is, by nature, only valid for frequencies less than about three times the spectral peak frequency. A higher order expansion is required to make the theory valid for higher frequencies. Such an expansion seems possible at first. But it is only to the third-order that wave fields can be statistically stationary and homogeneous without contradictions; energy transfer between waves does occur by nonlinear interaction in higher orders (see Hasselmann 1962). This energy transfer is reflected on the appearance of singularities when a higher order expansion is made along our line.

Although deep water is assumed throughout this paper, the theory is easily generalized to the case of finite depth. Practically, however, the case of deep water will be sufficient.

The present theory includes some assumptions and simplifications, which are to be examined *a posteriori*. Comparison of the theory with experiments will be made in detail in a later paper (Mitsuyasu, Kuo & Masuda 1979), where we will find some interesting features of wind waves arising from nonlinearity.

In summary we can say that the present theory provides a rather powerful means of analysing the nonlinear wind waves for frequencies less than about three times the spectral peak frequency. Other than the problem of the dispersion relation (cross spectra), at least three possible applications are pointed out.

(1) Comparison of the observed bispectra and the theory of Hasselmann, Munk & MacDonald (1963). The theoretical bispectra must be calculated from the spectra of free waves and not the observed spectra. The present method to separate the spectra of free and forced waves is useful.

(2) A phenomenon which is called overshoot. The magnitude of the second spectral peak may be predicted to a certain extent. Conspicuous examples of this phenomenon are expected when nonlinearity measured by $E_1\,\omega_m^4$ is strong as in a laboratory.

(3) The function which transfers the spectra of the pressure (or the velocity) in a definite depth to those of the surface displacement. So far, the frequency ω has been connected by the linear dispersion relation to the wavenumber which is at the same time the vertical decay rate of the wave motion. However, it is obvious that the transfer function thus determined gives erroneous results for frequencies where forced waves are larger than or comparable with free waves. Therefore separation of free and forced waves is indispensable to obtain the true transfer function for those frequencies.

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Appendix

Suppose that a dispersion relation changes by a small amount due to a certain cause, say, the nonlinearity or the change of the external fields. Then we may have the modified dispersion relation for fixed k:

$$C(k) \equiv \omega/k = C_0(k)+\epsilon g(k), \tag{A 1}$$

where $C_0(k)$ is the basic phase velocity and ϵ denotes a small quantity. That is, the increase of the phase velocity with k fixed is expressed as

$$\Delta C_k(k) = \epsilon g(k). \tag{A 2}$$

On the other hand, in the analysis of the experimental data it is much more convenient to rewrite (A 2) as the expression based on frequency. For that purpose we put

$$\omega = kC_0(k)+\epsilon kg(k) = k_0\, C_0(k_0), \tag{A 3}$$

where k_0 means the wavenumber when $\epsilon = 0$. Equations (A 1), (A 2) and (A 3) yield

$$\Delta C_\omega(\omega) \equiv \frac{\omega}{k}-\frac{\omega}{k_0} \doteqdot \frac{\Delta C_k(k_0)}{1+k_0(\ln C_0(k_0))'}, \tag{A 4}$$

the prime denoting the differentiation with respect to k. In particular for surface gravity waves $C_0(k) = k^{\frac{1}{2}}$, so that

$$\Delta C_\omega(\omega) \doteqdot 2\Delta C_k(k_0). \tag{A 5}$$

This relation explains the apparent difference of factor 2 between the result of Longuet-Higgins & Phillips (1962) and that of ours (2.24).

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